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ANALYTIC EXPRESSIONS FOR DERIVATIVES FROM SERIES SOLUTIONS TO THE THREE BODY PROBLEM

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Abstract

This paper presents a notation system to facilitate to solution of differential equations via Taylor series expansions and applies it to solve the circular restricted three body problem. Unlike previous Taylor series methods in the astrodynamics literature, computer algebra solvers are not used. Instead the notation system allows one to solve a system of differential equations analytically "by hand" without resorting to computer algebra software. This method produces recurrence relations explicitly in terms of a sequence of derivatives of the state with respect to time for the coefficients of Taylor Series solutions that can be evaluated numerically or manipulated further to investigate properties of the solution. For example, additional derivatives with respect to other parameters may also be found, including those that describe the dependence of the solution on initial conditions.

1 Introduction

In 1885, Acta Mathematica announced a prize in honor of King Oscar II of Sweden and Norway to anyone who could solve the n-body problem [1]. Poincaré won the prize, but not by solving the n-body problem. Instead Poincaré proved that the n-body problem could not be solved by reducing the dimension of the problem with integrals of the motion (as was done in the solution of the two-body problem). Over time, Poincaré's solution has led to what Florian Diacu [2] calls "folk mathematics", i.e the folk tale told by mathematicians, physicists, and astrodynamics that there is no solution to the 3-body or n-body problems. However, Poincaré only showed that one method couldn't be used to solve the 3-body or n-body problems. In fact, in 1912, Karl Sundman [3] found a solution to the three-body problem using power series that converged for all cases except those with collisions or zero initial angular momentum. The n-body problem took until 1991 until a similar solution was found by Quidong Wang [4]. However, although both Sundman's and Wang's solutions converge for most initial conditions, they converge very slowly and round off errors make them unusable in numerical work [2]. This has resulted in the larger astrodynamics community being mostly unaware of this method for solving the 3-body and n-body problem.

In this paper, I develop a general approach to solving ordinary differential equations like the 3-body problem and arrive at a result equivalent to Sundman's. However, this approach is much more easily generalized to solving other systems of differential equations via Taylor Series substitution. To do this, I focus on reducing the tedium of performing algebraic manipulations of power series when substituting them into a differential equation. I accomplish this with a new notation system that simplifies the manipulation of series expansions with nested summations, recursively-dependent coefficients, and other complexities. I then use this system to develop relations that facilitate addition, subtraction, multiplication, division, exponentiation, differentiation, and integration of power series. This then allows me to directly manipulate the series without resorting to either a computer algebra system or pages upon pages of nested sums and new variables for intermediate series expansions.

I then apply this method to solve the circular restricted three body problem with a recurrence relation for the coefficients of Taylor Series expansions of the state. Because I arrive at these expansions analytically, it is then possible to take derivatives of them with respect to the initial conditions or other parameters. Furthermore, this method can be applied to solve other challenging systems of differential equations.

2 Notation System

The notation system presented in this section is intended to facilitate working with complex summations over multiple indices. This will be used in the next section to develop methods for algebraic manipulation of power series.

2.1 Summation Indices

As in tensor algebra, indices will be both subscripted and superscripted. I.e., the expression: x^i, i shall denote an index rather than a power, i.e. the ith element of x as opposed to x to the ith power. A power of x will be denoted by parentheses, e.g. $(x)^i$. This allows the use of the Einstein summation convention where indices repeated both as subscripts and superscripts imply summation over that index:

$$a_n(x)^n = \sum_{n=0}^{\infty} a_n(x)^n = a_0 + a_1 x + a_2(x)^2 + \dots \tag{2.1}$$

This summation convention allows complicated sums to be written much more compactly:

$$\begin{aligned} a_n^i b_i c^j d_j (x)^n &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} a_n^i b_i \right) \left(\sum_{j=0}^{\infty} c^j d_j \right) (x)^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_n^i b_i c^j d_j (x)^n \end{aligned} \tag{2.2}$$

A repeated index is a placeholder for a sum (i.e. a "dummy index") and can be replaced without changing the meaning of a relation:

$$a_n^i b_i c^j d_j (x)^n = a_p^q b_q c^r d_r (x)^p \tag{2.3}$$

A convenient convention is to think of upstairs indices as rows of a matrix and downstairs as columns. This then allows easy transcription between this notation system and matrix notation, for example:

$$\mathbf{AB} = a_j^i b_k^j \tag{2.4}$$

$$\vec{x}^T \mathbf{A}\, \vec{x} = x_i a_j^i x^j \tag{2.5}$$

The Kronecker delta provides the Identity transformation, and is defined as:

$$\delta_{\rho_1 \rho_2 \dots \rho_m}^{\nu_1 \nu_2 \dots \nu_n} \equiv \begin{cases} 1 & \text{if all } \nu_n \text{ and } \rho_m \text{ equal} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

Where there are n upstairs indices and m downstairs indices.

We will also use the Levi-Civita symbol, which for n indices is defined as:

$$\epsilon^{\rho_1 \cdots \rho_n} \equiv \begin{cases} 1 & \text{if indices are an even permutation} \\ -1 & \text{if indices are an odd permutation} \\ 0 & \text{if any index is repeated} \end{cases} \tag{2.7}$$

For three indices the even permutations are: (1,2,3), (2,3,1), (3,1,2), and the odd permutations are: (1,3,2), (2,1,3), (3,2,1). The Levi-Civita symbol then allows us to write the vector cross product for arbitrary-dimension vectors:

$$\vec{c} = \vec{a} \times \vec{b} = c^i = \epsilon^{ijk} a_j b_k \tag{2.8}$$

2.1.1 Sigma

The Σ symbol, defined as:

$$\Sigma^{\rho_1\rho_2\ldots\rho_m}_{\mu_1\mu_2\ldots\mu_n} \equiv 1 \tag{2.9}$$

can be used indicate sums when indices are not repeated as subscripts and superscripts. For example, the geometric series may be described as:

$$\Sigma_n(-x)^n = 1 - x + (x)^2 - (x)^3 + \ldots \tag{2.10}$$

Since Σ is always equal to one, multiplying or dividing Σ with itself yields the following identities:

$$\Sigma^n\Sigma^n = \Sigma^n \tag{2.11}$$

$$\frac{\Sigma^n}{\Sigma^n} = \Sigma^n = \frac{1}{\Sigma^n} \tag{2.12}$$

When multiplying or dividing Σ coefficients with different indices, the indices may be combined:

$$\Sigma^m\Sigma^n\Sigma_p = \Sigma^{mn}_p \tag{2.13}$$

$$\frac{\Sigma_m}{\Sigma^n} = \Sigma^n_m = \frac{1}{\Sigma^n_m} \tag{2.14}$$

2.1.2 Range of Indices

When not specified, it will be assumed that indices can have any non-negative integer value and that sums run from zero to infinity. When other ranges are needed, we will say indices take on values from the set $\mathscr{I}$ (i.e. $i, j, k, etc. \in \mathscr{I}$) and define the set $\mathscr{I}$ used in a given case. To denote that indices could only have non-negative integer values, we may use: $\mathscr{I} \subseteq \mathbb{N}_0$. When needed, $\mathscr{I}_n$ can be used to denote values allowed by lowercase Latin indices, $\mathscr{I}_N$ for uppercase Latin indices, and $\mathscr{I}_\alpha$ for Greek indices. This allows us to specify limited ranges for special indices when we want to use some indices for infinite sums and others for finite ranges, for example: $\mathscr{I}_n = \mathbb{N}_0, \mathscr{I}_\alpha = \{0, 1, \ldots, 6\}$.

Sometimes it will be necessary to have some sums in an expression that start at different initial indices. For entities that have only one superscript or subscript, we maydenote a special starting index with a prepended superscript notation:

$$^{j}a_n \implies n \geq j \tag{2.15}$$

That is, in (2.15), n can take on values from $\mathscr{I}$ such that $n \geq j$. This restriction will affect anything summed with $^{j}a_n$, but not other terms. For example, consider the following (with $\mathscr{I} = \mathbb{N}_0$):

$$c_n = b^n a_n + b_n\,{}^{3}a_n \tag{2.16}$$

In (2.16), the first term is summed from 0 to ∞ and the second is summed from 3 to ∞. To denote a lower and an upper bound on an index, we may use two semicolon-separated prepended superscripts, like this:

$$^{j;k}a_n \implies j \leq n \leq k \tag{2.17}$$

And to put limits on a particular index when an entity has multiple indices, we may use a Σ symbol. For example, the following:

$$c_n =^{3;6}\Sigma^m a_{mn} \tag{2.18}$$

would sum a_{mn} over the first index from 3 to 6.

A principal utility of this notation is that it allows us to partition infinite sums when using the summation convention:

$$\begin{aligned} a_n(x)^n &= a_0 +^{1}a_n(x)^n \\ &= a_0 + a_1(x) +^{2}a_n(x)^n \\ &= a_0 + a_1(x) + a_2(x)^2 +^{3}a_n(x)^n \\ &= a_0 + a_1(x) +^{2;99}a_n(x)^n +^{100}a_n(x)^n \end{aligned} \tag{2.19}$$

This notation is also useful as a shorthand to specify a values for range of coefficients when defining a series. For example, the coefficients of an infinite series could be specified by the following:

$$a_0 = 2 \qquad a_1 = 0 \qquad {}^{2}a_n = 1/n \tag{2.20}$$

2.1.3 Compound-Index Notation and Implied Multiplication

In the definitions for the Kronecker delta and the Levi-Civita symbol in (2.6) and (2.7), I used a subscripted indices with ellipses such as "$\rho_1 \ldots \rho_n$" to indicate an arbitrary number of indices. This case happens often enough that a more convenient shorthand is needed. Compound indices inside of angle brackets will allow us to indicate a set of indices:

$$A^{\langle\rho_i\rangle} = A^{\rho_0\rho_1\cdots\rho_n} \tag{2.21}$$

In the above case, the subscript of the compound index is not assumed to be a member of the set $\mathscr{I}$ (i.e. a compound index's subscript does not have the same range as that index or other indices in an expression). Rather, the compound subscript is used to denote an arbitrary number of indices. If the compound subscript appears elsewhere in an expression, it will be assumed to run from 0 to the one less than the value of that index in other subscripts. E.g.: $a_i b^{<\rho_i>} = a_i b^{\rho_1\cdots\rho_{(i-1)}}$.

Although having $\langle\rho_i\rangle$ run from ρ_0 to $\rho_{(i-1)}$ instead of ρ_i is counter-intuitive, this convention pays off in simplifying complex relations that arise in later sections. This is primarily because we will want expressions like $\langle a^{\rho_i}\rangle$ to denote that a has i indices, but we will also want to use the i index elsewhere in an expression where indices start at 0. The only way to be consistent with both desires is to run i in $\langle a^{\rho_i}\rangle$ from 0 to $i-1$.

This compound-index notation also allows us to denote repeated multiplications when angle brackets are used on factors instead of indices. For example:

$$A^{\langle\rho_i\rangle}\langle b_{\rho_i}\rangle = A^{\rho_0\rho_1\cdots\rho_{(i-1)}} b_{\rho_0} b_{\rho_1} \ldots b_{\rho_{(i-1)}} \tag{2.22}$$

or:

$$A^{\langle\rho_i\rangle}\langle(b_{\rho_i})^i\rangle = A^{\rho_0\rho_1\cdots\rho_{(i-1)}} (b_{\rho_0})^0 (b_{\rho_1})^1 \ldots (b_{\rho_{(i-1)}})^{(i-1)} \tag{2.23}$$

This angle bracket shorthand allows definition of operations and symbols for arbitrary dimension. For example, the matrix determinant may be written for an $n \times n$ square matrix as:

$$\det\left|a^i_j\right| = \epsilon^{\langle\rho_i\rangle}\langle a^i_{\rho_i}\rangle = \epsilon^{\rho_0\rho_1\cdots\rho_{(n-1)}} a^0_{\rho_0} a^1_{\rho_1} \ldots a^{(n-1)}_{\rho_{(n-1)}} \tag{2.24}$$

where the range for each of the ρ_i indices goes from 0 to $n-1$. Notice that the i superscript of $a^i_{\rho_i}$ does not imply a summation with the i subscripts in the ρ_i indices. Instead, the index matching inside of angle brackets is used to describe the pattern in the implied multiplication.

We may also use a superscript outside of the angle brackets, analogous to a power, to denote when the same index is specific number of times:

$$A^{\langle\rho\rangle^s} = A^{\rho\rho\cdots\rho} \quad \text{where there are } s \text{ of the } \rho \text{ superscripts} \tag{2.25}$$

This outside superscript matches with the same index inside of the angle brackets. This allows for more complex indexing when needed, e.g.:

$$A^{\langle\rho_s\rangle^s} = A^{\langle\rho_0\rangle^0\langle\rho_1\rangle^1\langle\rho_2\rangle^2\cdots} = A^{\rho_1\rho_2\rho_2\rho_3\rho_3\rho_3\cdots} \tag{2.26}$$

In the case of nested angle brackets, the inner angle brackets are evaluated first:

$$\langle A^{\nu_j}_{\langle\rho_s\rangle}\rangle = \langle A^{\nu_j}_{\rho_0\cdots\rho_{(s-1)}}\rangle = A^{\nu_0}_{\rho_0\cdots\rho_{(s-1)}} A^{\nu_i}_{\rho_0\cdots\rho_{(s-1)}} \ldots A^{\nu_{(j-1)}}_{\rho_0\cdots\rho_{(s-1)}} \tag{2.27}$$

2.2 The Power Bracket Function & The Series Derivative

This section will introduce a modified power function, the *power bracket*, defined below, instead of the standard power function, $(x)^n$. This notation allows us to work more directly with the derivatives in Taylor series expansions and helps to reduce visual noise in complicated sums.

Definition 2.1. Let $x, \tilde{x} \in \mathbb{C}$ and $n \in \mathscr{I} \subseteq \mathbb{N}_0$, then the *power bracket* function is defined as:

$$[x]_n \equiv [x]^n \equiv \frac{1}{n!}(x-\tilde{x})^n \tag{2.28}$$

Where $\tilde{x}$ is the bracket power's *center point*. Alternatively, the center point can be explicitly specified as a second argument to the power bracket:

$$[x, \tilde{x}_i]_n \equiv [x, \tilde{x}_i]^n \equiv \frac{1}{n!}(x-\tilde{x}_i)^n \tag{2.29}$$

When $\tilde{x}$ is not specified it is assumed to be the same for all power brackets and other entities using the center point in a given expression.

$$f^n(\tilde{x})[x]_n = f^n(\tilde{x})[x,\tilde{x}]_n = \Sigma_n f^n(\tilde{x})\frac{1}{n!}(x-\tilde{x})^n \tag{2.30}$$

Usually there is no need to explicitly specify $\tilde{x}$, even when it is an argument to functions other than the bracket power.

$$f^n[x]_n + g^m[x]_m = f^n(\tilde{x})[x,\tilde{x}]_n + g^m(\tilde{x})[x,\tilde{x}]_m \tag{2.31}$$

Relations derived with $[x]^n$ or $[x]_n$ will hold for any allowable value of $\tilde{x}$. The goal in treating the center point as arbitrary is to allow series manipulations and the solution of a differential equation without choosing any particular point about which to expand the series until after the desired form of a series is found. I will formally justify this hidden variable approach in the next section.

The definition of the power bracket in (2.28) was chosen because it provides the following property when differentiated:

$$\begin{aligned}\frac{d}{dx}[x]_n &= \frac{n}{n!}(x-\tilde{x})^{n-1}\\ &= \frac{1}{(n-1)!}(x-\tilde{x})^{n-1}\\ &= [x]_{(n-1)}\end{aligned} \tag{2.32}$$

From (2.32), we see that the s-th order derivative of the power bracket is given by simply:

$$\frac{d^s}{dx^s}[x]_n = [x]_{(n-s)} \tag{2.33}$$

However, the price for this simple differentiation rule is that the multiplication of bracket powers of x is less straightforward than just adding exponents with the regular power function because of the $1/n!$ terms:

$$[x]_m[x]_n = \frac{1}{m!n!}(x-\tilde{x})^{(m+n)} = \frac{(m+n)!}{m!n!}[x]_{(m+n)} = \binom{m+n}{n}[x]_{(m+n)} \tag{2.34}$$

Similarly, the division of two power brackets of x is given by:

$$\frac{[x]_n}{[x]_m} = \frac{m!}{n!}\frac{(x-\tilde{x})^n}{(x-\tilde{x})^m} = \frac{m!(n-m)!}{n!}[x]_{(n-m)} = \frac{1}{\binom{n}{m}}[x]_{(n-m)} \tag{2.35}$$

The factorial terms must also be dealt with when raising to a power:

$$([x]_m)^n = \left(\frac{1}{m!}(x-\tilde{x})^m\right)^n = \frac{(mn)!}{(m!)^n}\frac{1}{(mn)!}(x-\tilde{x})^{(mn)} = \frac{(mn)!}{(m!)^n}[x]_{(mn)} \tag{2.36}$$

2.2.1 Differentiation and Integration of Series Using Bracket Powers

In the following sections I will derive relations for the manipulation of infinite series of bracket powers without regard for convergence. That is, I will treat them as formal power series.

Proposition 2.2. Consider the infinite series $a^n[x]_n$ with index $n \in \mathscr{I} \subseteq \mathbb{N}_0$ and each $a^n, x \in \mathbb{C}$. If the a^n are not functions of x, then the sth-derivative of the series with respect to x is given by:

$$\frac{d^s}{dx^s}a^n[x]_n = a^{(n+s)}[x]_n \tag{2.37}$$

Proof. $\frac{d^s}{dx^s}a_n[x]^n = \frac{d^s}{dx^s}\left({}^{0;k}a_n[x]^n + {}^{k+1}a_n[x]^n\right) = \frac{d^s}{dx^s}{}^{0;k}a_n[x]^n + \frac{d^s}{dx^s}{}^{k+1}a_n[x]^n$. The sth-derivative of the finite sum: ${}^{0;k}a_n[x]^n$ may be taken term by term and then summed. Since the a_n are not functions of x, by (2.32), the derivative of each term is ${}^{0;k}a_n[x]^{n-s}$ (here there is no summation and $n \geq s$). This may then be summed over n to give the sth-derivative of the sum, ${}^{0;k}a_n[x]^n$, as the sum, ${}^{s;k}a_n[x]^{n-s}$. We may then make the index substitution $m = n-s$ and write ${}^{s;k}a_n[x]^{n-s} = {}^{s;k}a_{(m+s)}[x]^m$. Since $(m+s) \geq s$ for all non-negative m, and since m is a dummy index and we are free to change its symbol, so: ${}^{s;k}a_{(m+s)}[x]^m = {}^{0;k}\Sigma_n a_{(n+s)}[x]^n$. Finally, since $\frac{d^s}{dx^s}a_n[x]^n = {}^{0;k}\Sigma_n a_{(n+s)}[x]^n + \frac{d^s}{dx^s}{}^{k+1}a_n[x]^n$ for all k, by induction then: $\frac{d^s}{dx^s}a_n[x]^n = a_{n+s}[x]^n$. $\square$

Proposition 2.3. Consider the infinite series $a^n[x]_n$ with index $n \in \mathscr{I} \subseteq \mathbb{N}_0$ and each $a^n, x \in \mathbb{C}$. If the a^n are not functions of x, then the sth indefinite integral over x of the series is given by:

$$\int \dots \int a^n[x]_n (dx)^s = {}^{0,s-1}A^n[x]_n + a^n[x]_{(n+s)} \tag{2.38}$$

where the ${}^{0,s-1}A_n$ are constants of integration.

Proof. By Proposition 2.2: $\frac{d^s}{dx^s} a_{n-s}[x]^n = a_n[x]^n$. Therefore, by the Fundamental Theorem of Calculus, $\int \dots \int a_n[x]^n (dx)^s = a_{n-s}[x]^n$. $\square$

In Proposition 2.3 we run into a subtle consequence of the allowed index ranges. In (2.38) the index n is restricted to $\mathscr{I} \subseteq \mathbb{N}_0$. If we subtracted s from the subscript for the integral, e.g. $a^{(n-s)}$, then we imply that the index could take on negative values outside of $\mathscr{I}$. To avoid this, Proposition 2.3 puts the $n+s$ on the bracket power and partitions the sum with the ${}^{0,s-1}A^n$ coefficients for the constants of the integration.

2.2.2 Series Derivative

If we construct a Taylor series such that the point about which the series is expanded is a free parameter, then we can make a univariate function from C^∞ into a bivariate function that describes all possible Taylor series expansions of that function. We can also use bracket powers to simply the Taylor series expansion so that the f^n coefficients are explicitly the nth-derivatives of $f(x)$ evaluated at $\tilde{x}$. I will call this Taylor series with the more explicit use of derivatives as series coefficients and the hidden center point, the *series derivative*:

Definition 2.4. Let $\mathscr{I} \subseteq \mathbb{N}_0$, $x, \tilde{x} \in X \subseteq \mathbb{C}$, and $f(x) \in C^\infty(X)$ (i.e., $f(x)$ is infinitely differentiable throughout X), then the *series derivative* of $f(x)$ is:

$$f^n(\tilde{x})[x]_n = f^n[x]_n \tag{2.39}$$

where:

$$f^n(\tilde{x}) = f^n = \frac{d^n}{dx^n} f(x)\bigg|_{x=\tilde{x}} \tag{2.40}$$

We may choose to explicitly show $\tilde{x}$ in a series derivative as $f^n(\tilde{x})[x]_n$. Or we may omit it and use $f^n[x]_n$ with the convention that all center points in a given expression are consistent. We can keep $\tilde{x}$ hidden away until we need it. The $\tilde{x}$ doesn't affect the derivatives or algebraic operations we perform on a series derivative.

Proposition 2.5. Consider the series derivative $f^n(\tilde{x})[x]_n$ with index $n \in \mathscr{I} \subseteq \mathbb{N}_0$. Then the sth-derivative of the series respect to $\tilde{x}$ is:

$$\frac{d^s}{d\tilde{x}^s} f^n(\tilde{x})[x]_n = 0 \tag{2.41}$$

Proof. Since $f^n(\tilde{x}) = \frac{d^n}{dx^n} f(x)\big|_{x=\tilde{x}}$ then $\frac{d}{d\tilde{x}} f^n(\tilde{x}) = f^{(n+1)}$. But $\frac{d}{d\tilde{x}}[x]_n = -[x]_{n-1}$ and $\frac{d}{d\tilde{x}} f^n(\tilde{x})[x]_n = f^{(n+1)}(\tilde{x})[x]_n - {}^1f^n(\tilde{x})[x]_{n-1} = f^{(n+1)}(\tilde{x})[x]_n - f^{(n+1)}(\tilde{x})[x]_n = 0$. Since $\frac{d}{d\tilde{x}} f^n(\tilde{x})[x]_n = 0$, all higher derivatives are also zero. $\square$

If the starting function is analytic everywhere with a radius of convergence large enough to cover the whole domain of the starting function, then the bivariate function from the Taylor series is always equivalent to the univariate function for any expansion center point, $\tilde{x}$. But what if the radius of convergence is smaller? What if the function is somewhere non-analytic and the radius of convergence is zero? Well, even then there is an $\tilde{x}$ for any x that lets us get $f(x)$ from the series derivative:

Theorem 2.6. $\exists(\tilde{x}, x) \in X : \frac{d^s}{dx^s} f^n(\tilde{x})[x]_n = \frac{d^s}{dx^s} f(x), \forall s \in \mathbb{N}_0$

Proof. If $\tilde{x} = x$, then by Prop. 2.2, $\frac{d^s}{dx^s} f^n(\tilde{x})[x]_n\big|_{x=\tilde{x}} = f^s = \frac{d^s}{dx^s} f(x)$ $\square$

For any x there is at least one $\tilde{x}$ that returns values for $f(x)$ and all of its derivatives from $f^n(\tilde{x})[x]_n$. This means we can manipulate a functions series derivative like a formal power series, without regard to convergence, and at the very least, we will have relations valid for the function's derivatives at $\tilde{x} = x$ for any x in the function's domain.

2.3 Superscripts vs. Subscripts

In the following section most results will hold for either subscripted indices or superscripted indices. However, in a given expression it is important to maintain the location of an index when performing algebraic manipulations. This is both to preserve any implied summations but also because, as in tensor notation, there will be an implied connection between the index location and the type of differentiation performed. We can view a sum over superscripted indices as a series derivative with total derivatives and subscripted indices suggest a connection to a partial derivative. I.e., $a^n[t]_n$ suggests that each a^n is the n-th total derivative of $a(t)$ with respect to t and $a_n[x]^n$ suggests each a_n is $\frac{\partial^n}{\partial x^n}a(x)$.

3 Algebraic Manipulation of Series Derivatives

This section builds a unified framework that will facilitate the algebraic manipulation of series derivatives and to make it more convenient than previous approaches to the algebraic manipulation of Taylor series expansions. Although the application here is with series derivatives and bracket powers, all of these methods could be re-written for the manipulation of generic power series. In the following, relations for algebraic manipulation of entities with upstairs indices also hold for downstairs indices and vice versa. However, this isn't always explicitly shown to avoid unnecessary tedium. (I.e., I have tried to limit myself to just necessary tedium.)

3.1 Addition and Subtraction

Due to the distributive law of elementary algebra, series derivatives may be added and subtracted term by term by combining the coefficients of like terms:

$$f(x) + g(x) = f^m[x]_m + g^n[x]_n = (f^m + g^m)[x]_m \tag{3.1}$$

$$f(x) - g(x) = f^m[x]_m - g^n[x]_n = (f^m - g^m)[x]_m \tag{3.2}$$

3.2 Multiplication

Proposition 3.1. Let $\mathscr{I} \subseteq \mathbb{N}_0$, then two differential series may be multiplied using the binomial coefficient, $\binom{p}{m} = p!/(n!(n-m)!)$, with this relation:

$$(f^m[x]_m)(g^n[x]_n) = \Sigma_m \binom{p}{m} f^m g^{(p-m)}[x]_p \tag{3.3}$$

Proof. When we multiply $(f_m[x]^m)(g_n[x]^n)$, $[x]^m[x]^n = \binom{m+n}{m}[x]^{m+n}$ in each term of the sum. Therefore: $(f_m[x]^m)(g_n[x]^n) = \binom{m+n}{m} f_m g_n [x]^{m+n}$. We may then make the change of indices $p = m+n$ to arrive at (3.3). □

The above is merely a restatement of the Cauchy product for power series in terms of bracket powers. But in this form, it also reminds us of the general Leibniz Rule for the pth order derivative of the product of $f(x)$ and $g(x)$. It is also evocative of the binomial theorem.

Let's explore this connection with the binomial series further. We start by introducing the β-coefficient as:

$$\beta^n_{pq} = \begin{cases} \binom{n}{p} & \text{if } p+q = n \\ 0 & \text{if } p+q \neq n \end{cases} \tag{3.4}$$

Then:

$$\beta^n_{pq} f^p g^q = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \begin{cases} \binom{n}{p} f^p g^q & \text{if } p+q = n \\ 0 & \text{if } p+q \neq n \end{cases} \tag{3.5}$$

and since $p + q = n$ for nonzero terms, we may substitute $q = n - p$:

$$\beta^n_{pq} f^p g^q = \sum_{p=0}^{\infty} \binom{n}{p} f^p g^{(n-p)} = \Sigma_p \binom{n}{p} f^p g^{(n-p)} \tag{3.6}$$

This then enables an alternate form of (3.3):

$$(f^p[x]_p)(g^q[x]_q) = \beta^n_{pq} f^p g^q [x]_n \tag{3.7}$$

If we instead define this β-coefficient for an arbitrary number of indices using the multinomial coefficient $(\frac{(\rho_0+\rho_1+\ldots+\rho_{(i-1)})!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!})$, we can then use it to multiply an arbitrary number of differential series together.

Definition 3.2. Let $\mathscr{I} \subseteq \mathbb{N}_0$, then the β-coefficient is defined as:

$$\beta^{\langle\rho_i\rangle}_n \equiv \beta^n_{\langle\rho_i\rangle} \equiv \begin{cases} \frac{n!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!} & \text{if } \Sigma^i \rho_i = n \\ 0 & \text{if } \Sigma^i \rho_i \neq n \end{cases} \tag{3.8}$$

Theorem 3.3. *For $\mathscr{I} \subseteq \mathbb{N}_0$, the product of an arbitrary number of bracket series is given by:*

$$(f^0_{\rho_0}[x]^{\rho_0})(f^1_{\rho_1}[x]^{\rho_1})\ldots(f^{(i-1)}_{\rho_{(i-1)}}[x]^{\rho_{(i-1)}}) = \beta^{\langle\rho_i\rangle}_n \langle f^i_{\rho_i}\rangle [x]^n \tag{3.9}$$

or:

$$(f^{\rho_0}_0[x]_{\rho_0})(f^{\rho_1}_1[x]_{\rho_1})\ldots(f^{\rho_{(i-1)}}_{(i-1)}[x]_{\rho_{(i-1)}}) = \beta^n_{\langle\rho_i\rangle} \langle f^{\rho_i}_i\rangle [x]_n \tag{3.10}$$

Proof. First we prove (3.9). From the definition of the bracket power in (2.28), the product of i bracket powers is:

$$\begin{aligned} [x]^{\rho_0}[x]^{\rho_1}\ldots[x]^{\rho_{(i-1)}} &= \frac{1}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}(x-\tilde{x})^{\rho_0+\rho_1+\ldots+\rho_{(i-1)}} \\ &= \frac{(\rho_0+\rho_1+\ldots+\rho_{(i-1)})!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}[x]^{\rho_0+\rho_1+\ldots+\rho_{(i-1)}} \end{aligned} \tag{3.11}$$

where $\frac{(\rho_0+\rho_1!+\ldots+\rho_{(i-1)})!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}$ is the multinomial coefficient for the ρ_i. If we wish to replace the $[x]^{\rho_0+\rho_1+\ldots+\rho_{(i-1)}}$ with $[x]^n$, we could do that unless we were also summing over the ρ_i. But this is exactly what we are doing with the $f^i_{\rho_i}$ when we multiply $(f^0_{\rho_0}[x]^{\rho_0})(f^1_{\rho_1}[x]^{\rho_1})\ldots(f^{(i-1)}_{\rho_{(i-1)}}[x]^{\rho_{(i-1)}})$. Because of this, when we write each $[x]^n$ term we must ensure that it is multiplied by all combinations of the $f^i_{\rho_i}$ such that the ρ_i sum to n and that each product of the $f^i_{\rho_i}$ have the proper multinomial coefficient. Summing with $\beta^{\langle\rho_i\rangle}_n$ as it is defined in Definition 3.2 achieves this:

$$\begin{aligned} f^0_{\rho_0}[x]^{\rho_0}\ldots f^{(i-1)}_{\rho_{(i-1)}}[x]^{\rho_{(i-1)}} &= f^0_{\rho_0}\ldots f^{(i-1)}_{\rho_{(i-1)}}\frac{(\rho_0+\ldots+\rho_{(i-1)})!}{\rho_0!\ldots\rho_{(i-1)}!}[x]^{\rho_0+\ldots+\rho_{(i-1)}} \\ &= \begin{cases} f^0_{\rho_0}\ldots f^{(i-1)}_{\rho_{(i-1)}}\frac{n!}{\rho_0!\ldots\rho_{(i-1)}!}[x]^n & \text{if } \Sigma^i\rho_i = n \\ 0 & \text{if } \Sigma^i\rho_i \neq n \end{cases} \\ &= \beta^{\langle\rho_i\rangle}_n\langle f^i_{\rho_i}\rangle[x]^n \end{aligned} \tag{3.12}$$

The above proof also holds for (3.10) when repeated with superscripted and subscripted indices swapped. ☐

3.2.1 Properties of the β-coefficients

These β-coefficients have many interesting properties that will be useful will when deriving relations in the following sections.

First, the β-coefficient with two indices, β^n_m, reduces to the Kronecker delta.

Identity 3.4. If $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta^n_m = \delta^n_m \tag{3.13}$$

Proof. Per Definition 3.2, if β^n_m is interpreted as $\beta^n_{\langle\rho_m\rangle}$:

$$\beta^n_m = \begin{cases} \frac{n!}{m!} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{3.14}$$

Since $\frac{n!}{m!} = 1$ when $n = m$, this is equivalent to the Kronecker Delta, δ^n_m. The same also holds if the if β^n_m is interpreted as $\beta^{\langle\rho_n\rangle}_n$. ☐

The β-coefficient multiplication identities below for the $\beta^n_{\langle\rho_m\rangle}$ form also hold for the $\beta_n^{\langle\rho_m\rangle}$ form.

Identity 3.5. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta_q^{nj}\beta_n^{\langle\rho_m\rangle}\beta_j^{\langle\nu_i\rangle} = \beta_q^{\langle\rho_m\rangle\langle\nu_i\rangle} \tag{3.15}$$

Proof. The left side is only nonzero when all of the β-coefficients in the product are nonzero. This happens when $\Sigma^m\rho_m = n$, $\Sigma^i\nu_i = j$, and $n+j=q$. I.e., when $\Sigma^m\rho_m + \Sigma^i\nu_i = q$. In those cases the product of $\beta_q^{nj}\beta_n^{\langle\rho_m\rangle}\beta_n^{\langle\rho_m\rangle}\beta_j^{\langle\nu_i\rangle}$ is $\frac{q!}{n!j!}\frac{n!}{\rho_0!\rho_1!\ldots\rho_{(m-1)}!}\frac{j!}{\nu_1 0!\nu_1!\ldots\nu_{(i-1)}!}$, which, per Definition 3.2, is equivalent to $\beta_q^{\langle\rho_m\rangle\langle\nu_i\rangle}$. □

Identity 3.6. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta_n^{j\langle\rho_m\rangle}\beta_j^{\langle\nu_i\rangle} = \beta_n^{\langle\rho_m\rangle\langle\nu_i\rangle} \tag{3.16}$$

Proof. By Definition 3.2, $n = j + \Sigma^m\rho_m = \Sigma^i\nu_i + \Sigma^m\rho_m$ for the β-coefficients to be non-zero, and $\frac{n!}{j!\rho_0!\rho_1!\ldots\rho_{(m-1)}!}\frac{j!}{\nu_0!\nu_1!\ldots\nu_{(i-1)}!} = \frac{n!}{\rho_0!\rho_1!\ldots\rho_{(m-1)}!\nu_0!\nu_1!\ldots\nu_{(i-1)}!}$ □

Identity 3.7. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta_n^{\langle\rho_m\rangle}\langle\beta_{\rho_m}^{\langle\nu_i\rangle}\rangle = \beta_n^{\langle\nu_i\rangle^m} \tag{3.17}$$

Proof. Recall that $\beta_n^{\langle\rho_m\rangle}\langle\beta_{\rho_m}^{\langle\nu_i\rangle}\rangle$ denotes $\beta_n^{\rho_1\cdots\rho_{(m-1)}}(\beta_{\rho_0}^{\langle\nu_i\rangle}\beta_{\rho_1}^{\langle\nu_i\rangle}\ldots\beta_{\rho_{(m-1)}}^{\langle\nu_i\rangle})$. So $\Sigma^m\rho_m = n$ for $\beta_n^{\langle\rho_m\rangle} \neq 0$, and for each $\beta_{\rho_m}^{\langle\nu_i\rangle} \neq 0$ the $\Sigma^i\nu_i = \rho_m$. This means $(\Sigma^i\nu_i)^m = n$ is required for the expression in (3.17) to be nonzero. In that case it is equal to $\frac{n!}{\rho_0!\ldots\rho_{(m-1)}!}\frac{\rho_0!}{\nu_0!\ldots\nu_{(i-1)}!}\cdots\frac{\rho_{(m-1)}!}{\nu_1!\ldots\nu_{(i-1)}!} = \frac{n!}{(\nu_1!)^m\ldots(\nu_{(i-1)}!)^m} = \beta_n^{\langle\nu_i\rangle^m}$ □

3.3 Raising to a Positive Integer Power

We can use Theorem 3.3 to develop relations for raising a differential series to a power. Let's start with a definition:

Definition 3.8. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $x, z \in \mathbb{C}$, then:

$$(f^n[x]_n)^z = f^{n\bullet(z)}[x]_n \tag{3.18}$$

or

$$(f_n[x]^n)^z = f_n^{\bullet(z)}[x]^n \tag{3.19}$$

where $f^{n\bullet(z)}$ and $f_n^{\bullet(z)}$ are the *dot powers* of f^n and f_n respectively.

For dot powers with exponents in $\mathbb{N}_0$, we can use β-coefficients with implied multiplication to write an expression for these dot power coefficients:

Proposition 3.9. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $m \in \mathbb{N}_0$, then:

$$f^{n\bullet(m)} = \beta^n_{\langle\rho_m\rangle}\langle f^{\rho_m}\rangle \tag{3.20}$$

and

$$f_n^{\bullet(m)} = \beta_n^{\langle\rho_m\rangle}\langle f_{\rho_m}\rangle \tag{3.21}$$

Proof. We may raise a series to a positive integer power by performing successive multiplications on the series using Theorem 3.3:

$$(f^n[x]_n)^m = \beta^n_{\langle\rho_m\rangle}\langle f^{\rho_m}\rangle[x]_n = f^{n\bullet(m)} = f^n_{\bullet(m)} \tag{3.22}$$

and

$$(f_n[x]^n)^m = \beta_n^{\langle\rho_m\rangle}\langle f_{\rho_m}\rangle[x]^n = f_n^{\bullet(m)} = f_{n\bullet(m)} \tag{3.23}$$

□

In the following sections, for brevity, I will present relations for dot powers using only the $f_n^{\bullet(m)}$ form, but these also hold for other forms.

Now let's derive a few useful identities. As with scalar powers, we can add positive integer dot powers when multiplying:

Identity 3.10. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s, t \in \mathbb{N}_0$, then:

$$\beta_q^{mn} a_m^{\bullet(s)} a_n^{\bullet(t)} = a_q^{\bullet(s+t)} \tag{3.24}$$

Proof. Employing Proposition 3.9 with Identity 3.5:

$$\begin{aligned} \beta_q^{mn} a_m^{\bullet(s)} a_n^{\bullet(t)} [x]^q &= \beta_q^{mn} \beta_m^{\langle\rho_s\rangle} \langle a_{\rho_s} \rangle \beta_n^{\langle\nu_t\rangle} \langle a_{\nu_t} \rangle [x]^q \\ &= \beta_q^{\langle\rho_s\rangle\langle\nu_t\rangle} \langle a_{\rho_s} \rangle \langle a_{\nu_t} \rangle [x]^q \\ &= \beta_q^{\langle\gamma_{s+t}\rangle} \langle a_{\gamma_{s+t}} \rangle [x]^q \\ &= a_q^{\bullet(s+t)} [x]^q \end{aligned} \tag{3.25}$$

By comparing like terms in $\beta_q^{mn} a_m^{\bullet(s)} a_n^{\bullet(t)} [x]^q = a_q^{\bullet(s+t)} [x]^q$, we arrive at (3.24). □

The dot powers of zero and one behave as expected:

Identity 3.11. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$a_m^{\bullet(1)} = a_m \tag{3.26}$$

Proof. By Proposition 3.9: $a_m^{\bullet(1)} = \beta_n^m a_n$. By Identity 3.4: $\beta_n^m a_n = \delta_n^m a_n = a_m$. □

Identity 3.12.

$$a_0^{\bullet(0)} = 1 \quad \text{and} \quad {}^1a_n^{\bullet(0)} = 0 \tag{3.27}$$

Proof. By Identities 3.10 and 3.11: $\beta_q^{mn} a_m^{\bullet(0)} a_n^{\bullet(1)} = a_q^{\bullet(0+1)} = a_q$. But by Identity 3.11: $\beta_q^{mn} a_m^{\bullet(0)} a_n = a_q$. This can be true for all x iff $a_0^{\bullet(0)} = 1$ and ${}^1a_n^{\bullet(0)} = 0$. □

Remark 3.13. Identities 3.10 and 3.11 provide a recursive formula for calculating dot-powers. Setting $t = p-1$ and $s = 1$ in (3.24) yields the following relation for $p \in \mathbb{N}_0$ and $\mathscr{I} \subseteq \mathbb{N}_0$:

$$a_q^{\bullet(p)} [x]^q = \beta_q^{nm} a_n a_m^{\bullet(p-1)} [x]^q \tag{3.28}$$

3.3.1 Binomial Expansions of Dot-Powers

We may use the pre-pended superscript notation to break apart a sum into two parts, for example:

$$a_n [x]^n = a_0 + {}^1a_n [x]^n \tag{3.29}$$

We may then raise $a_n [x]^n$ to the positive power s using a binomial expansion:

$$\begin{aligned} (a_n [x]^n)^s &= (a_0 + {}^1a_n [x]^n)^s \\ &= {}^{0,s}\Sigma_m \binom{s}{m} (a_0)^m ({}^1a_n [x]^n)^{s-m} \\ &= {}^{0,s}\Sigma_m \binom{s}{m} (a_0)^m \; {}^1a_n^{\bullet(s-m)} [x]^n \\ &= \beta_{ij}^s (a_0)^i \; {}^1a_n^{\bullet(j)} [x]^n \end{aligned} \tag{3.30}$$

Now let's generalize this to ${}^ka_n^{\bullet(s)}$:

Lemma 3.14. *Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s \in \mathbb{N}_0$, then:*

$$^ka_n^{\bullet(s)} = \beta_{ij}^s \beta_n^{q\langle r\rangle^i} \delta_k^r (a_k)^i \; {}^{(k+1)}a_q^{\bullet(j)} \tag{3.31}$$

Proof. Applying the Binomial Theorem as in (3.30):

$$\begin{aligned} ({}^k a_n[x]^n)^s &= (a_k[x]_k +^{(k+1)} a_q[x]^q)^s \\ &= {}^{0,s}\Sigma_i \binom{s}{i} (a_k[x]_k)^i ({}^{(k+1)}a_q[x]^q)^{s-i} \end{aligned} \tag{3.32}$$

We then simplify with the formulas for raising bracket powers to a power, (2.36), and for multiplying bracket powers, (2.34):

$$\begin{aligned} ({}^k a_n[x]^n)^s &= {}^{0,s}\Sigma_i \frac{s!}{i!(s-i)!} \frac{(ki)!}{(k!)^i} (a_k)^i [x]_{(ki)} \ {}^{(k+1)}a_q^{\bullet(s-i)}[x]^q \\ &= {}^{0,s}\Sigma_i \frac{s!}{i!(s-i)!} \frac{(ki)!}{(k!)^i} \frac{(ki+q)!}{(ki)!q!} (a_k)^i \ {}^{(k+1)}a_q^{\bullet(s-i)} \Sigma^q [x]_{(ki+q)} \\ &= {}^{0,s}\Sigma_i \frac{s!}{i!(s-i)!} \frac{(ki+q)!}{(k!)^i q!} (a_k)^i \ {}^{(k+1)}a_q^{\bullet(s-i)} \Sigma^q [x]_{(ki+q)} \end{aligned} \tag{3.33}$$

Next we replace indices with $n = ki + q$ and $j = s - i$ and introduce β-coefficients:

$$({}^k a_n[x]^n)^s = \Sigma^q \beta^s_{ij} \beta^n_{q\langle k\rangle^i} (a_k)^i \ {}^{(k+1)}a_q^{\bullet(j)}[x]_n \tag{3.34}$$

There are only sums in i, j, q and n above, not in k or s. In order to keep the n as a subscript to match the left hand side, while still avoiding a sum in k, we may use the Kronecker Delta, δ^r_k from (2.6):

$$({}^k a_n[x]^n)^s = \beta^s_{ij} \beta^{q\langle r\rangle^i}_n \delta^r_k (a_k)^i \ {}^{(k+1)}a_q^{\bullet(j)}[x]_n \tag{3.35}$$

□

Next we need the following identities that will allow us to simplify the ${}^k a_n^{\bullet(s)}$ type factors.

Identity 3.15. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s \in \mathbb{N}_0$, then:

$${}^k a_n^{\bullet(s)} = 0 \quad \text{if } n < ks \tag{3.36}$$

Proof. The definition of a dot-power, from (3.20), is: $a_n^{\bullet(s)} = \beta_n^{\langle\rho_m\rangle} a_{\rho_s}$. Each $a_n^{\bullet(s)}$ term is comprised of a sum of β-products of the a_{ρ_s} coefficients such that s of the ρ_s indices sum to n. If the smallest allowed value for the ρ_s is k, then the smallest value for this sum is ks. Therefore we must have $n \geq ks$ for $a_n^{\bullet(s)}$ to be nonzero. □

Identity 3.16. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s \in \mathbb{N}_0$, then:

$${}^k a_n^{\bullet(s)} = \delta^r_k \beta_n^{\langle r\rangle^s} (a_k)^s \quad \text{if } n = ks \tag{3.37}$$

Proof. Since there are no subscripts of ${}^k a_n$ less than k, the only way to sum s subscripts to ks is to have s factors of $a_k[x]_k$ multiplied together: $(a_k[x]_k)^s = \frac{(ks)!}{(k!)^s}(a_k)^s[x]_{(ks)} = {}^k a_{(ks)}^{\bullet(s)}[x]_{(ks)}$. The δ^r_k in (3.37) is used to prevent summation over k. □

3.3.2 Rational and Complex Powers

We can also use binomial expansions for rational and complex number powers using Newton's generalized binomial coefficient:

Definition 3.17. Given $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, then *Newton's Binomial Coefficient* is given by:

$$\binom{z}{n} = \frac{(z)_n}{n!} \tag{3.38}$$

where $(z)_n$ is the descending Pochhammer symbol, representing:

$$(z)_n = z(z-1)(z-2)\dots(z-n+1) \tag{3.39}$$

This then allows binomial expansions for any power $z \in \mathbb{C}$:

Theorem 3.18. *Given $\mathscr{I} \subseteq \mathbb{N}_0$ and the a_n,$z \in \mathbb{C}$, if $a_0 \neq 0$ then:*

$$a_n^{\bullet(z)} = \Sigma_s \binom{z}{s} (a_0)^{(z-s)}\ {}^1a_n^{\bullet(s)} \tag{3.40}$$

Proof. We start by writing $(a_n[x]^n)^z$ in binomial form:

$$(a_n[x]^n)^z = (a_0 +^1 a_n[x]^n)^z \tag{3.41}$$

Let $y(x) =^1 a_n[x]^n$. If we take the s-th derivative of the left hand side with respect to y, we get:

$$\frac{d^s}{dy^s}(a_0 +^1 a_n[x]^n)^z = (z)_s(a_0 +^1 a_n[x]^n)^{(z-s)} \tag{3.42}$$

The Taylor series expansion of $(a_n[x]^n)^z$ in y with $\tilde{y} = 0$ is:

$$\begin{aligned}(a_n[x]^n)^z &= \Sigma_s \frac{(z)_s}{s!}(a_0)^{(z-s)}(y)^s \\ &= \Sigma_s \binom{z}{s}(a_0)^{(z-s)}({}^1a_n[x]^n)^s \\ &= \Sigma_s \binom{z}{s}(a_0)^{(z-s)}\ {}^1a_n^{\bullet(s)}[x]^n \end{aligned} \tag{3.43}$$

where $\binom{z}{s}$ is Newton's Binomial Coefficient from Definition 3.17. Note that we need $a_0 \neq 0$ when $z \notin \mathbb{N}_0$ because the exponent of $(a_0)^{(z-s)}$ in (3.40) can go negative if $z \notin \mathbb{N}_0$. $\square$

In Theorem 3.18, (3.40) is a finite sum in s because ${}^1a_n^{\bullet(s)} = 0$ when $s > n$. We can now take a series, $a_n[x]^n$ to any rational, real, or complex power when $a_0 \neq 0$. And even when $a_0 = 0$, we could shift the center point (provided a(x) is not identically 0), and then apply Theorem 3.18 to the shifted series where $a_0 \neq 0$.

For cases where there are multiple possible values, the $(a_0)^{(z-s)}$ factor will determine which branch is generated by the series. I.e., if $(a_0)^{(z-s)}$ is multi-valued there will be a different series for each branch of $(a_0)^{(z-s)}$.

4 The Circular Restricted 3-Body Problem

Now let's use the tools from Section 3 to attack the Circular Restricted 3-Body Problem. In this section we will develop a recurrence relation between the derivatives of the position for a test mass in the gravitational field created by two bodies in a circular orbit.

4.1 Nondimensionalization of the Problem

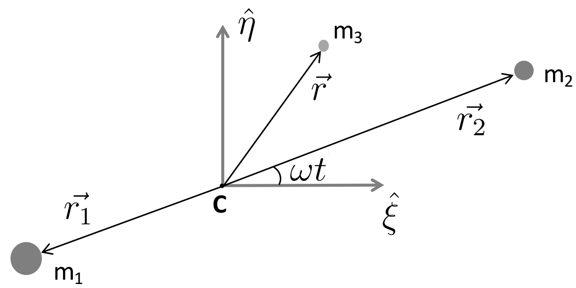


Figure 1: The Circular Restricted 3-Body Problem (CR3BP)

Figure 1 depicts the geometry of the problem. Here point **C** is the center of mass of the system and $\vec{r_1} = -r_1\hat{r}$ and $\vec{r_2} = r_2\hat{r}$ describe the position of the bodies m_1 and m_2 relative to the center of mass, where $\hat{r} = \cos(\omega t)\hat{\xi} + \sin(\omega t)\hat{\eta}$. I will nondimensionalize the problem by assuming units such that $r_1 + r_2 = 1$ and $\mu_1 + \mu_2 = 1$ (where μ_1 and μ_2 are the masses m_1 and m_2 times Newton's Gravitational Constant). Since $\vec{r_1}$ and $\vec{r_2}$ are with respect to the center of mass:

$$m_1\vec{r_1} + m_2\vec{r_2} = (-m_1r_1 + m_2r_2)\hat{r_2} = 0 = (-\mu_1r_1 + \mu_2r_2)\hat{r_2} \tag{4.1}$$

Noting that $r_2 = 1 - r_1$ and $\mu_1 = 1 - \mu_2$:

$$-\mu_1r_1 + \mu_2r_2 = (\mu_2 - 1)r_1 + \mu_2(1 - r_1) = 0 \tag{4.2}$$

When we solve (4.2) for r_1, we get:

$$r_1 = \mu_2 = \nu \tag{4.3}$$

Here ν is a parameter that describes both the relative positions and relative masses of the two gravitating bodies.

Finally, note that if we assume the two-bodies are in a circular orbit, this nondimensionalization yields:

$$\omega = \frac{1}{r_1 + r_2}\sqrt{\frac{\mu_1 + \mu_2}{r_1 + r_2}} = 1 \tag{4.4}$$

4.2 Lagrangian Equations of Motion in Rotating Frame

To derive the Lagrangian Equations of Motion, we start by defining a rotating reference frame (remember $\hat{r_2}$ is from **C** to m_2):

$$\hat{x} = \hat{r_2} \qquad \hat{z} = \hat{\xi} \times \hat{\eta} \qquad \hat{y} = \hat{z} \times \hat{x} \tag{4.5}$$

The position of m_3 relative the center of mass is then:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{4.6}$$

And $\vec{\rho_1}$ and $\vec{\rho_2}$ are vectors to body 3 from bodies 1 and 2 respectively:

$$\vec{\rho_1} = (x + \nu)\hat{x} + y\hat{y} + z\hat{z} \tag{4.7}$$

$$\vec{\rho_2} = (x + \nu - 1)\hat{x} + y\hat{y} + z\hat{z} \tag{4.8}$$

with magnitudes:

$$\rho_1^2 = (x + \nu)^2 + (y)^2 + (z)^2 \tag{4.9}$$

$$\rho_2^2 = (x + \nu - 1)^2 + (y)^2 + (z)^2 \tag{4.10}$$

The specific potential energy, U, of m_3 is:

$$U = -\frac{1 - \nu}{\rho_1} - \frac{\nu}{\rho_2} \tag{4.11}$$

and the specific kinetic energy, T, of m_3 is:

$$T = \frac{1}{2}(\dot{\vec{r_I}} \cdot \dot{\vec{r_I}}) \tag{4.12}$$

and $\dot{\vec{r_I}}$ is the inertial-frame velocity and is given by:

$$\dot{\vec{r_I}} = \dot{\vec{r}} + \vec{\omega} \times \vec{r} = (\dot{x} - y)\hat{x} + (\dot{y} + x)\hat{y} + \dot{z}\hat{z} \tag{4.13}$$

We may then write T as:

$$T = \frac{1}{2}\left[(\dot{x} - y)^2 + (\dot{y} + x)^2 + (\dot{z})^2\right] \tag{4.14}$$

The Lagrangian, $L = T - U$, is then:

$$L = \frac{1}{2}\left[(\dot{x} - y)^2 + (\dot{y} + x)^2 + (\dot{z})^2\right] + \frac{1 - \nu}{\rho_1} + \frac{\nu}{\rho_2} \tag{4.15}$$

We can now get the equations of motion from the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \tag{4.16}$$

We start with the $\frac{\partial L}{\partial q_i}$:

$$\frac{\partial L}{\partial x} = (\dot{y} + x) - \frac{1-\nu}{(\rho_1)^3}(x+\nu) - \frac{\nu}{(\rho_2)^3}(x+\nu-1) \tag{4.17}$$

$$\frac{\partial L}{\partial y} = -(\dot{x} - y) - \frac{1-\nu}{(\rho_1)^3}y - \frac{\nu}{(\rho_2)^3}y \tag{4.18}$$

$$\frac{\partial L}{\partial z} = -\frac{1-\nu}{(\rho_1)^3}z - \frac{\nu}{(\rho_2)^3}z \tag{4.19}$$

The $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$ are then:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}(\dot{x} - y) = \ddot{x} - \dot{y} \tag{4.20}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}(\dot{y} + x) = \ddot{y} + \dot{x} \tag{4.21}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{d}{dt}\dot{z} = \ddot{z} \tag{4.22}$$

So the equations of motion in the rotating frame are:

$$\ddot{x} = 2\dot{y} + x - \frac{1-\nu}{(\rho_1)^3}(x+\nu) - \frac{\nu}{(\rho_2)^3}(x+\nu-1) \tag{4.23}$$

$$\ddot{y} = -2\dot{x} + y - y\frac{1-\nu}{(\rho_1)^3} - y\frac{\nu}{(\rho_2)^3} \tag{4.24}$$

$$\ddot{z} = -z\frac{1-\nu}{(\rho_1)^3} - z\frac{\nu}{(\rho_2)^3} \tag{4.25}$$

4.3 Series Solution to the Lagrangian Equations of Motion

Let's begin the solution by introducing series derivatives of $(\rho_1)^2$ and $(\rho_2)^2$:

$$a_1(t) = (\rho_1)^2 = a_1^n[t]_n \qquad\qquad a_2(t) = (\rho_2)^2 = a_2^n[t]_n \tag{4.26}$$

That will then let us substitute in the series derivatives for the x, y, and z that we hope to solve for:

$$\begin{aligned} a_1^n[t]_n &= (\nu)^2 + 2\nu x + (x)^2 + (y)^2 + (z)^2 \\ &= (\nu)^2 + 2\nu x^n[t]_n + x^{n\bullet(2)}[t]_n + y^{n\bullet(2)}[t]_n + z^{n\bullet(2)}[t]_n \\ &= \left[(\nu)^2 + 2\nu x^n + x^{n\bullet(2)} + y^{n\bullet(2)} + z^{n\bullet(2)}\right][t]_n \end{aligned} \tag{4.27}$$

$$\begin{aligned} a_2^n[t]_n &= (\nu-1)^2 + 2(\nu-1)x + (x)^2 + (y)^2 + (z)^2 \\ &= (\nu-1)^2 + 2(\nu-1)x^n[t]_n + x^{n\bullet(2)}[t]_n + y^{n\bullet(2)}[t]_n + z^{n\bullet(2)}[t]_n \\ &= \left[(\nu-1)^2 + 2(\nu-1)x^n + x^{n\bullet(2)} + y^{n\bullet(2)} + z^{n\bullet(2)}\right][t]_n \end{aligned} \tag{4.28}$$

We may then manipulate these series derivatives so that they can be substituted more easily into the equations of motion:

$$\frac{1-\nu}{(\rho_1)^3} = (1-\nu)a_1^{n\bullet(-\frac{3}{2})}[t]_n \qquad\qquad \frac{\nu}{(\rho_2)^3} = \nu a_2^{n\bullet(-\frac{3}{2})}[t]_n \tag{4.29}$$

Now, let's start with the equation of motion in x, with some rearrangement:

$$\ddot{x}(t) = 2\dot{y}(t) + x(t) - (1-\nu)(a_1(t))^{-3/2}(x+\nu) - \nu(a_2(t))^{-3/2}(x(t)+\nu-1) \tag{4.30}$$

The series derivative form is then:

$$\begin{aligned} x^{(n+2)}[t]_n &= 2y^{n+1}[t]_n + x^n[t]_n - \left((1-\nu)a_1^{p\bullet(-\frac{3}{2})}[t]_p + \nu a_2^{p\bullet(-\frac{3}{2})}[t]_p\right) x^q[t]_q \\ &\quad +(\nu^2-\nu)\left(a_1^{n\bullet(-\frac{3}{2})}[t]_n - a_2^{n\bullet(-\frac{3}{2})}[t]_n\right) \\ &= \left[2y^{n+1} + x^n - \beta^n_{pq}\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})} + \nu a_2^{p\bullet(-\frac{3}{2})}\right) x^q + (\nu^2-\nu)\left(a_1^{n\bullet(-\frac{3}{2})} - a_2^{n\bullet(-\frac{3}{2})}\right)\right][t]_n \end{aligned} \tag{4.31}$$

Next the equation of motion in y:

$$\ddot{y} = -2\dot{x}(t) + y(t) - (1-\nu)(a_1(t))^{-3/2}y(t) - \nu(a_2(t))^{-3/2}y(t) \tag{4.32}$$

$$\begin{aligned} y^{(n+2)}[t]_n &= -2x^{n+1}[t]_n + y^n[t]_n - \left((1-\nu)a_1^{p\bullet(-\frac{3}{2})}[t]_p + \nu a_2^{p\bullet(-\frac{3}{2})}[t]_p\right) y^q[t]_q \\ &= \left[-2x^{n+1} + y^n - \beta^n_{pq}\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})} + \nu a_2^{p\bullet(-\frac{3}{2})}\right) y^q\right][t]_n \end{aligned} \tag{4.33}$$

Finally the equation of motion in z:

$$\ddot{z} = -(1-\nu)(a_1(t))^{-3/2}z(t) - \nu(a_2(t))^{-3/2}z(t) \tag{4.34}$$

$$\begin{aligned} z^{(n+2)}[t]_n &= -\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})}[t]_p + \nu a_2^{p\bullet(-\frac{3}{2})}[t]_p\right) z^q[t]_q \\ &= -\beta^n_{pq}\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})} + \nu a_2^{p\bullet(-\frac{3}{2})}\right) z^q[t]_n \end{aligned} \tag{4.35}$$

We now have recurrence relations for the derivatives of x,y, and z:

$$x^{(n+2)} = 2y^{n+1} + x^n - \beta^n_{pq}\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})} + \nu a_2^{p\bullet(-\frac{3}{2})}\right) x^q + (\nu^2-\nu)\left(a_1^{n\bullet(-\frac{3}{2})} - a_2^{n\bullet(-\frac{3}{2})}\right) \tag{4.36}$$

$$y^{(n+2)} = -2x^{n+1} + y^n - \beta^n_{pq}\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})} + \nu a_2^{p\bullet(-\frac{3}{2})}\right) y^q \tag{4.37}$$

$$z^{(n+2)} = -\beta^n_{pq}\left((1-\nu)a_1^{p\bullet(-\frac{3}{2})} + \nu a_2^{p\bullet(-\frac{3}{2})}\right) z^q \tag{4.38}$$

$$a_1^p = (\nu)^2 + 2\nu x^p + x^{p\bullet(2)} + y^{p\bullet(2)} + z^{p\bullet(2)} \tag{4.39}$$

$$a_2^p = (\nu-1)^2 + 2(\nu-1)x^p + x^{p\bullet(2)} + y^{p\bullet(2)} + z^{p\bullet(2)} \tag{4.40}$$

Here the $x^{(n+2)}, y^{(n+2)}$, and $z^{(n+2)}$ are only functions of $x^{(1+2)}, y^{(1+2)}, z^{(1+2)}$, and lower derivatives. In addition, all of the sums in the recurrence relations above are finite. The derivatives are valid for any center point and can be used wherever $x(t), y(t)$, and $z(t)$ have derivatives.

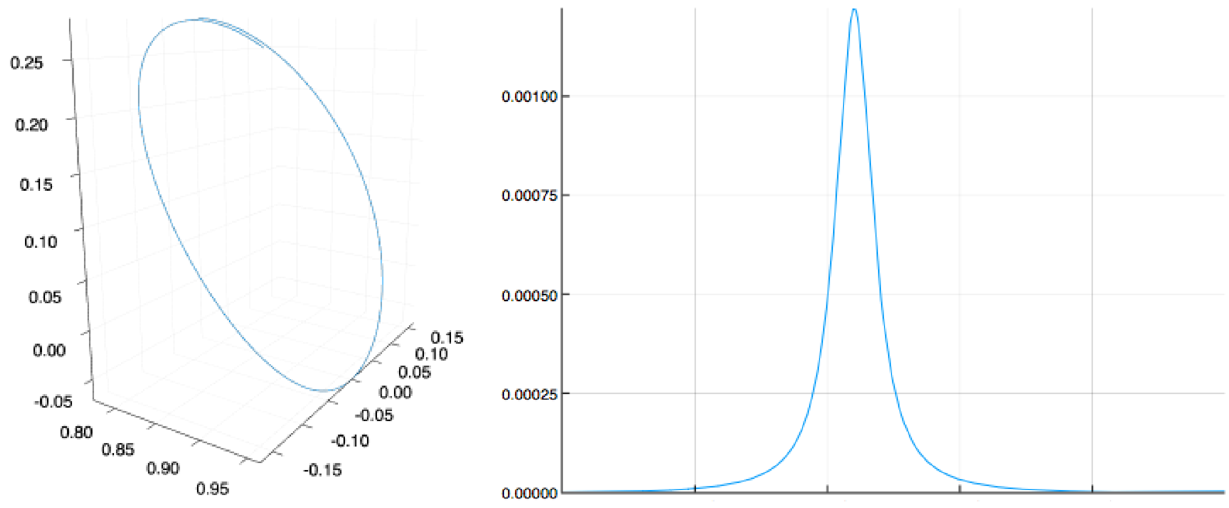
4.4 Evaluation of this Solution

To evaluate this solution numerically, we truncate the series at some order and then take steps to control he error. The coefficients in the recurrence relations are recomputed at each step using the results from the previous step. As an example for numerical evaluation, I chose an L1 halo orbit from K.C. Howell's 1984 paper on halo orbits [5]. Specifically, I used the following initial condition (in the rotating frame) with $\nu = 0.04$:

$$x_0 = \begin{bmatrix} 0.777413 & 0 & 0.284268 & 0 & 0.361870 & 0 \end{bmatrix} \tag{4.41}$$

This was solve with a 5th order expansion and a fixed step size of 0.0002 (in non-dimensional time). Figure 2 shows the result of the integration for a little over one period of the halo orbit (2.5 non-dimensional time). We can see the shape of the halo, but it did not return to the initial point, and if we were to calculate another revolution, it would diverge from the halo. The variation of the Jacobi integral from its initial value over the orbit is also shown.

Better error control than the results in Figure 2 could easily be achieved with any number of numerical integration methods with a much faster run time. But that's not the point of this method. What we have is a symbolic representation of the solution to the circular restricted three body problem from which we may make mathematical mischief.



(a) plot of one revolution

(b) variation of Jacobi Intergral

Figure 2: Example L1 halo orbit calculation with order 5 series and step of 0.0002

5 Derivatives from this Solution

The recurrence relations for the series derivatives of x,y, and z give us all of the time derivatives, but how do we get other derivatives? More specifically, how to we get the derivatives of each term in the series derivative with respect to some other parameter. Such derivatives could then allow us to convert $x_i(t)$ functions into functions of t and some new parameter, λ, using a multivariate Taylor Series:

$$x_i(t,\lambda) = \frac{d^s x_i^n}{(d\lambda)^s}[t]_n[\lambda]_s \tag{5.1}$$

There are two approaches for finding these derivatives. The first approach is to take the derivative of the ODE that we're solving with respect to some new parameter and then treat the resulting equation as a new ODE to solve. For example the first derivatives of the CR3BP equations of motion with respect to a $\lambda \neq \nu$ are:

$$\frac{d\ddot{x}}{d\lambda} = 2\frac{d\dot{y}}{d\lambda} + \frac{dx}{d\lambda} + \frac{3}{2}\frac{1-\nu}{(a_1)^{5/2}}(x+\nu)\frac{da_1}{d\lambda} - \frac{1-\nu}{(a_1)^{3/2}}\frac{dx}{d\lambda} + \frac{3}{2}\frac{\nu}{(a_2)^{5/2}}(x+\nu-1)\frac{da_2}{d\lambda} - \frac{\nu}{(a_2)^{3/2}}\frac{dx}{d\lambda} \tag{5.2}$$

$$\frac{d\ddot{y}}{d\lambda} = -2\frac{d\dot{x}}{d\lambda} + \frac{dy}{d\lambda} + \frac{3}{2}\frac{1-\nu}{(a_1)^{5/2}}y\frac{da_1}{d\lambda} - \frac{1-\nu}{(a_1)^{3/2}}\frac{dy}{d\lambda} + \frac{3}{2}\frac{\nu}{(a_2)^{5/2}}y\frac{da_2}{d\lambda} - \frac{\nu}{(a_2)^{3/2}}\frac{dy}{d\lambda} \tag{5.3}$$

$$\frac{d\ddot{z}}{d\lambda} = \frac{3}{2}\frac{1-\nu}{(a_1)^{5/2}}z\frac{da_1}{d\lambda} - \frac{1-\nu}{(a_1)^{3/2}}\frac{dz}{d\lambda} + \frac{3}{2}\frac{\nu}{(a_2)^{5/2}}z\frac{da_2}{d\lambda} - \frac{\nu}{(a_2)^{3/2}}\frac{dz}{d\lambda} \tag{5.4}$$

$$\frac{da_1}{d\lambda} = 2(x+\nu)\frac{dx}{d\lambda} + 2y\frac{dy}{d\lambda} + 2z\frac{dz}{d\lambda} \tag{5.5}$$

$$\frac{da_2}{d\lambda} = 2(x+\nu-1)\frac{dx}{d\lambda} + 2y\frac{dy}{d\lambda} + 2z\frac{dz}{d\lambda} \tag{5.6}$$

which then results in the following recurrence relations:

$$
\begin{aligned}
\frac{dx^{(n+2)}}{d\lambda} &= 2\frac{dy^{(n+1)}}{d\lambda} + \frac{dx^n}{d\lambda} + \frac{3}{2}\beta^n_{pqr}\left((1-\nu)a_1^{p\bullet(\text{-}5/2)}(x^q+\nu)\frac{da_1^r}{d\lambda} + \nu a_2^{p\bullet(\text{-}5/2)}(x^q+\nu-1)\frac{da_2^r}{d\lambda}\right) \qquad (5.7)\\
&\quad -\beta^n_{pq}\left((1-\nu)a_1^{p\bullet(\text{-}3/2)}\frac{dx^q}{d\lambda} + \nu a_2^{p\bullet(\text{-}3/2)}\frac{dx^q}{d\lambda}\right)\\
\frac{dy^{(n+2)}}{d\lambda} &= -2\frac{dx^{(n+1)}}{d\lambda} + \frac{dy^n}{d\lambda} + \frac{3}{2}\beta^n_{pqr}\left((1-\nu)a_1^{p\bullet(\text{-}5/2)}y^q\frac{da_1^r}{d\lambda} + \nu a_2^{p\bullet(\text{-}5/2)}y^q\frac{da_2^r}{d\lambda}\right) \qquad (5.8)\\
&\quad -\beta^n_{pq}\left((1-\nu)a_1^{p\bullet(\text{-}3/2)}\frac{dy^q}{d\lambda} + \nu a_2^{p\bullet(\text{-}3/2)}\frac{dy^q}{d\lambda}\right)\\
\frac{dz^{(n+2)}}{d\lambda} &= \frac{3}{2}\beta^n_{pqr}\left((1-\nu)a_1^{p\bullet(\text{-}5/2)}z^q\frac{da_1^r}{d\lambda} + \nu a_2^{p\bullet(\text{-}5/2)}z^q\frac{da_2^r}{d\lambda}\right) \qquad (5.9)\\
&\quad -\beta^n_{pq}\left((1-\nu)a_1^{p\bullet(\text{-}3/2)}\frac{dz^q}{d\lambda} + \nu a_2^{p\bullet(\text{-}3/2)}\frac{dz^q}{d\lambda}\right)\\
\frac{da_1^n}{d\lambda} &= \beta^n_{pq}\left(2(x^p+\nu)\frac{dx^q}{d\lambda} + 2y^p\frac{dy^q}{d\lambda} + 2z^p\frac{dz^q}{d\lambda}\right) \qquad (5.10)\\
\frac{da_2^n}{d\lambda} &= \beta^n_{pq}\left(2(x^p+\nu-1)\frac{dx^q}{d\lambda} + 2y^p\frac{dy^q}{d\lambda} + 2z^p\frac{dz^q}{d\lambda}\right) \qquad (5.11)
\end{aligned}
$$

The second approach is to use the chain rule on the series derivative directly:

$$
\frac{dx_i^n}{d\lambda} = {}^{1;n}\Sigma^k\Sigma^j\,\frac{\partial x_i^n}{\partial x_j^{(n\text{-}k)}}\frac{dx_j^{(n\text{-}k)}}{d\lambda} \qquad (5.12)
$$

where the partials: $\partial x_i^n/\partial x_j^{(n\text{-}k)}$ are from the recurrence relation for the x_i^n. This can be generalized to higher order derivatives:

$$
\frac{d^s x_i^n}{(d\lambda)^s} = {}^{1;n}\Sigma^k\left(\frac{\langle\rho_j!\rangle}{s!}\beta^s_{\langle\rho_j\rangle}\left\langle\frac{d^{\rho_j}x_j^n}{(d\lambda)^{\rho_j}}\left\langle\frac{\partial}{\partial x_j^{(n\text{-}k)}}\right\rangle^{\rho_j}\right\rangle\right)x_i^{(n\text{-}k)} \qquad (5.13)
$$

where j counts through the different x_i (e.g., x, y, and z). This approach can be an easier approach to get higher derivatives, if we can get the $\partial x_i^n/\partial x_j^{(n\text{-}k)}$ partials from the recurrence relation.

In the case of the CR3BP recurrence relations, most of the $\partial x_i^n/\partial(x_j^{(n-1)})$ are zero, except for these two:

$$
\begin{aligned}
\frac{\partial x^n}{\partial(y^{(n\text{-}1)})} &= 2 \qquad (5.14)\\
\frac{\partial y^n}{\partial(x^{(n\text{-}1)})} &= -2 \qquad (5.15)
\end{aligned}
$$

We may then ease the task of finding the $\partial x_i^n/\partial(x_j^{(n-2)})$ partials by adding a_1 and a_2 as x_i coordinates along with x,y, and z. This then gives us:

$$
\begin{aligned}
\frac{\partial x^n}{\partial(x^{(n\text{-}2)})} &= 1-(1-\nu)a_1^{n\bullet(\text{-}3/2)} - \nu a_2^{n\bullet(\text{-}3/2)} \qquad (5.16)\\
\frac{\partial y^n}{\partial(y^{(n\text{-}2)})} &= 1-(1-\nu)a_1^{n\bullet(\text{-}3/2)} - \nu a_2^{n\bullet(\text{-}3/2)} \qquad (5.17)\\
\frac{\partial z^n}{\partial(z^{(n\text{-}2)})} &= 1-(1-\nu)a_1^{n\bullet(\text{-}3/2)} - \nu a_2^{n\bullet(\text{-}3/2)} \qquad (5.18)\\
\frac{\partial x^n}{\partial(a_1^{(n\text{-}2)})} &= \frac{3}{2}(1-\nu)(x+\nu)a_1^{n\bullet(\text{-}5/2)} \qquad (5.19)\\
\frac{\partial x^n}{\partial(a_2^{(n\text{-}2)})} &= \frac{3}{2}\nu(x+\nu-1)a_2^{n\bullet(\text{-}5/2)} \qquad (5.20)
\end{aligned}
$$

$$\frac{\partial y^n}{\partial(a_1^{(n\text{-}2)})} = \frac{3}{2}(1-\nu)ya_1^{n\bullet(\text{-}5/2)} \tag{5.21}$$

$$\frac{\partial y^n}{\partial(a_2^{(n\text{-}2)})} = \frac{3}{2}\nu ya_2^{n\bullet(\text{-}5/2)} \tag{5.22}$$

$$\frac{\partial z^n}{\partial(a_1^{(n\text{-}2)})} = \frac{3}{2}(1-\nu)za_1^{n\bullet(\text{-}5/2)} \tag{5.23}$$

$$\frac{\partial z^n}{\partial(a_2^{(n\text{-}2)})} = \frac{3}{2}\nu za_2^{n\bullet(\text{-}5/2)} \tag{5.24}$$

where the the $da_1^n/d\lambda$ and $da_2^n/d\lambda$ are given by (5.10) and (5.11). All other $\partial x_i^n/\partial x_j^{(n\text{-}k)}$ partials zero. Furthermore, for higher order derivatives, only $\partial^s x_i^n/(\partial x_j^{(n\text{-}k)})^s$ partials involving a_1 and a_2 are nonzero. (I won't include these derivatives here, but they are straightforward from the above first derivatives.)

Such derivatives, from either method above, allow us to convert x,y, and z from functions of t alone to functions of t and λ using a multivariate Taylor Series. One application of this would be. to convert the $x(t)$, $y(t)$, and $z(t)$ into functions of the components of the initial velocity u, v, and w. For that case, the recurrence relations above would be started from $\frac{dx^1}{du} = 1$, $\frac{dy^1}{dv} = 1$, $\frac{dz^1}{dw} = 1$ with the other initial $d/d\lambda$'s zero.

6 Conclusion

The notation system and methods presented in this paper provide a new method for the solution of systems of differential equations such as the equations of motion for the circular restricted three body problem. These solutions may be slow to evaluate numerically, but they are represented symbolically and can be manipulated to study the behavior of the solution. For example, the solution can be re-formulated to depend explicitly on additional parameters such as the initial conditions. While numerical integration techniques can approximate a particular solution to a set of differential equations, these methods provide a general solution. Although these solutions can only be evaluated in an approximate sense as a truncated series, their infinite series forms do represent a general solution for the system. Furthermore, by Theorem 2.6 this solution is valid for the derivatives of the solution *even when the Taylor Series doesn't converge.*

The notation system and series manipulation methods presented in this paper allows this approach to be readily adapted to solve other difficult systems of ordinary differential equations. In future papers, I will use this method to present solutions to the n-body problem, the motion of a test particle in an arbitrary force field, and Euler's laws of motion for a rigid body.

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